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The Saint-Venant problem of the bending of a cylinder with helical anisotropy☆

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ABSTRACT

The Saint-Venant problems of pure bending and the bending of a cylinder with helical anisotropy by a transverse force are reduced to boundary-value problems for systems of ordinary differential equations with variable coefficients. The problems are solved by two methods – the small-parameter method and numerical methods. The behaviour of the stiffnesses and the stress-strain state is investigated as a function of the parameters of the problem.

The Saint-Venant problems of stretching twisting for a cylinder with helical anisotropy was investigated in Refs. 1–5. Below we consider the case of bending when the principal moment and the principal stress vector in the cross-section are orthogonal to the cylinder axis.

1. Fundamental relations of the theory of elasticity for a body with helical anisotropy and formulation of the problem

Consider a cylindrical body of length L, which occupies a volume $V = S \times [0, L]$, where S is a ring with inner radius r_1 and outer radius r_2. We connect the origin of a Cartesian system of coordinates x_1, x_2, x_3 with the geometrical centre of one of the cross-sections of the cylinder. In addition to the Cartesian system we also introduce a helical system of coordinates r, θ, z, connected with the first system by the relations

$$x_1 = r\cos(\theta + \tau z), \quad x_2 = r\sin(\theta + \tau z), \quad x_3 = z; \quad \tau = \text{const} \tag{1.1}$$

When $r = \text{const}$ and $\theta = \text{const}$ relations (1.1) define a helical curve, the pitch of which $h = 2\pi/\tau$. We will represent the radius vector of points of the helical curve in the form

$$\mathbf{R} = r\mathbf{e}_1' + z\mathbf{e}_3'$$

Here

$$\mathbf{e}_1' = \mathbf{i}_1\cos(\theta + \tau z) + \mathbf{i}_2\sin(\theta + \tau z), \quad \mathbf{e}_2' = -\mathbf{i}_1\sin(\theta + \tau z) + \mathbf{i}_2\cos(\theta + \tau z)$$

We connect a Frenet reference frame with the helical curve; $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are the unit vectors of the principal normal, binormal and tangent. Using the formulae

$$\frac{d\mathbf{R}}{ds} = \mathbf{e}_3, \quad \frac{d\mathbf{t}}{ds} = k\mathbf{e}_1, \quad \mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$$

$$ds = g\,dz, \quad g^2 = 1 + x^2, \quad x = \tau r$$

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where $k=\tau^2 r/g^2$ is the curvature of the helical curve, after reduction we obtain an orthogonal transition matrix from basis $\mathbf{e}_j$ to basis $\mathbf{e}_i'$

$$A = \left\| \begin{matrix} -1 & 0 & 0 \\ 0 & -1/g & x/g \\ 0 & x/g & 1/g \end{matrix} \right\|$$

As previously,[3] we will assume that the material of the cylinder is locally transversely isotropic, the direction of the axis of symmetry of which is defined by the vector $\mathbf{e}_3$. With this assumption, in the basis $\mathbf{e}_i$ Hooke's law has the form

$$\sigma_i = c_{ij}\varepsilon_j, \quad i, j = 1, \dots, 6$$

$$\sigma_k = \sigma_{kk}, \quad \sigma_4 = \sigma_{23}, \quad \sigma_5 = \sigma_{13}, \quad \sigma_6 = \sigma_{12}$$

$$\varepsilon_k = \varepsilon_{kk}, \quad \varepsilon_4 = 2\varepsilon_{23}, \quad \varepsilon_5 = 2\varepsilon_{13}, \quad \varepsilon_6 = 2\varepsilon_{12}; \quad k = 1, 2, 3$$

where σ_{ij} are the components of the stress tensor, ε_{ij} are the components of the tensor of small deformations, while the moduli c_{ij}, expressed in terms of the technical constants[6] E, E', G', ν, ν', have the form

$$c_{11} = c_{22} = \frac{E(E' - E\nu'^2)}{\gamma(1+\nu)}, \quad c_{33} = \frac{E'^2(1-\nu)}{\gamma}, \quad c_{66} = \frac{E}{2(1+\nu)}$$
$$c_{12} = \frac{E(E'\nu + E\nu'^2)}{\gamma(1+\nu)}, \quad c_{13} = c_{23} = \frac{EE'\nu'}{\gamma}, \quad \gamma = E'(1-\nu) - 2E\nu'^2$$
$$c_{15} = c_{16} = c_{25} = c_{26} = c_{35} = c_{36} = 0, \quad c_{44} = c_{55} = G' \tag{1.2}$$

Expressions for the moduli c_{ij}' in the helical system of coordinates were given previously in Ref. 4.

The components of the strain tensor in the basis of the helical system of coordinates are expressed in terms of the displacements u_r, u_θ, u_z by the following formulae

$$e_{rr} = \partial_r u_r, \quad e_{\theta\theta} = \frac{1}{r}(u_r + \partial_\theta u_\theta), \quad e_{zz} = Du_z$$
$$2e_{r\theta} = \partial_r u_\theta + \frac{1}{r}(\partial_\theta u_r - u_\theta), \quad 2e_{rz} = \partial_r u_z + Du_r, \quad 2e_{z\theta} = \partial_\theta u_z + Du_\theta \tag{1.3}$$

The equilibrium equations in the stresses in this case have the form

$$\partial_r(r\sigma_{rr}) - \sigma_{\theta\theta} + \partial_\theta\sigma_{r\theta} + rD\sigma_{rz} = 0, \quad \partial_r(r\sigma_{r\theta}) + \sigma_{r\theta} + \partial_\theta\sigma_{\theta\theta} + rD\sigma_{\theta z} = 0,$$
$$\partial_r(r\sigma_{rz}) + \partial_\theta\sigma_{\theta z} + rD\sigma_{zz} = 0 \tag{1.4}$$

Here

$$\partial_r = \frac{\partial}{\partial r}, \quad \partial_\theta = \frac{\partial}{\partial\theta}, \quad \partial = \frac{\partial}{\partial z}, \quad D = \partial - \tau\partial_\theta$$

We will assume that the side surface of the cylinder is stress-free, i.e.,

$$r = r_\alpha \ (\alpha = 1, 2): \sigma_{rr} = 0, \quad \sigma_{r\theta} = 0, \quad \sigma_{rz} = 0$$

Introducing the displacement vector $\mathbf{u} = (u_r, u_\theta, u_z)^T$, we can represent the problem in the following vector-operator form

$$M(\partial, \tau)\mathbf{u} \equiv \partial^2 A_0\mathbf{u} + \partial A_1\mathbf{u} + A_2\mathbf{u} = 0 \tag{1.5}$$

$$N(\partial, \tau)\mathbf{u} \equiv (\partial B_0\mathbf{u} + B_1\mathbf{u})|_\Gamma = 0 \tag{1.6}$$

Here A_k and B_i are matrix differential operators in the variables r and θ, and the values of the indices $k=0, 1, 2$ and $i=0, 1$ indicate the order of these operators; we will not give their specific form here in view of their complexity and since the method of constructing them is obvious. We will merely note that the coefficients of these operators depend on r and θ, but are independent of z, which enables us to seek the solution in the form

$$\mathbf{u} = \mathbf{a}e^{\gamma z}$$

As a result, we obtain an eigenvalue problem in the section $z=\text{const}$

$$M_1(\gamma)\mathbf{a} \equiv \{M(\gamma)\mathbf{a}, N(\gamma)\mathbf{a}\} = 0 \tag{1.7}$$

It is known from the general theory of quadratic beams of symmetrical operators,[7] that the spectrum of the operator $M_1(\gamma)$ is discrete, has a point of condensation at infinity and is situated symmetrically in the complex plane $\gamma = \alpha + i\beta$, i.e., for any eigenvalue

$$\gamma_s^+ = \gamma_s = \alpha_s + i\beta_s, \quad \alpha_s \ge 0, \quad \beta_s \ge 0$$

when $\alpha \neq 0$ there are three more eigenvalues

$$\gamma_{-s}^{\pm} = \alpha - i\beta, \quad \gamma_s^- = -\gamma_s, \quad \gamma_{-s}^- = -\alpha - i\beta$$

It was shown in Refs. 1–4 that $\gamma_0 = 0$, $\gamma_1^{\pm} = \pm i\tau$ are four-fold eigenvalues and, apart from $\gamma_1^{\pm}$, there are no other pure imaginary eigenvalues. Hence, in a general representation, the solutions of problem (1.7) are

$$\mathbf{u} = \mathbf{u}_S + \mathbf{u}_P$$

where $\mathbf{u}_S$ is the Saint-Venant solution, corresponding to the eigenvalues and, $\mathbf{u}_P$ is the solution corresponding to the remaining part of the spectrum and has the form

$$\mathbf{u}_P = \sum_k [C_k^- \mathbf{u}_k^-(z) + C_k^+ \mathbf{u}_k^+(z - L)]; \quad \mathbf{u}_k^{\pm}(z) = \mathbf{a}_k^{\pm} \exp(\gamma_k^{\pm} z)$$

where $C_k^{\pm}$ are arbitrary constants. The solution $\mathbf{u}_S$ covers the whole region, and the solution $\mathbf{u}_P$ is localized in the region of the ends of the cylinder $z = 0, L$ and decreases exponentially with distance from them. The rate of decrease is determined by the parameter $\alpha_* = \inf(\alpha_k)$, the value of which depends on the degree of anisotropy. The ratio E/E' can serve, in particular, as the characteristic of the degree of anisotropy of an orthotropic material, which, in composite fibre materials with a soft filling, may be considerably less than unity, which means that α_* is small. It must be emphasised that the stress-strain state for the Saint-Venant solution in any cross-section $z = \text{const}$ is equivalent in an integral sense to the stress-strain state corresponding to the principal vector and the principal moment of the external forces, applied to one of the ends of the cylinder. The principal vector and the principal moment of the stresses, corresponding to any vector function $u_k^{\pm}(z)$, are equal to zero.[4]

2. Elementary solutions of Saint-Venant bending problems

The solution of Saint-Venant bending problems[3,4] is a linear combination of the elementary solutions corresponding to the eigenvalues $\gamma_1^{\pm} = \pm i\tau$ and can be represented in the form

$$\mathbf{u}_S = 2\mathrm{Re}\left(\sum_{l=1}^{4} C_l \mathbf{u}_l\right), \quad \mathbf{u}_l = e^{i\psi} \sum_{k=1}^{l} \frac{z^{l-k}}{(l-k)!} \mathbf{a}_k$$
$$\mathbf{a}_1 = (1, i, 0)^T, \quad \mathbf{a}_2 = (0, 0, -r)^T, \quad \mathbf{a}_3 = (a_{r,3}, ia_{\theta,3}, ia_{z,3})^T, \quad \mathbf{a}_4 = (ia_{r,4}, -a_{\theta,4}, -a_{z,4})^T \tag{2.1}$$

Here C_l are arbitrary constants, which are determined when the boundary conditions on the ends of the cylinder are satisfied. Note that the stress-strain state, corresponding to the first two elementary solutions, are identically equal to zero, since $u_{x_1}^0 = 2Re(C_1)$ and $u_{x_2}^0 = -2Im(C_1)$ are the displacements and $\omega_{x_1} = 2Im(C_2)$, $\omega_{x_2} = Re(C_2)$ are small angles of rotation of the cylinder as a rigid body; the stress-strain state corresponding to $\mathrm{Re}(C_3u_3)$ and $\mathrm{Im}(C_3u_3)$, is equivalent, in the integral sense, to the stress-strain state corresponding solely to the bending moments M_{x_1}, $-M_{x_2}$; the stress-strain state corresponding to $\mathrm{Re}(C_4u_4)$ and $\mathrm{Im}(C_4u_4)$ is equivalent to the stress-strain state corresponding to the transverse forces Q_{x_1}, $-Q_{x_2}$ and the bending moments.

We will introduce the stress vector $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_6)^T$. The stress vectors corresponding to the elementary solutions $\mathbf{a}_s (s = 3, 4)$, can be represented in the form

$$\boldsymbol{\sigma}_s = e^{i\psi}\mathbf{b}_s, \quad \psi = \theta + \tau z$$
$$\mathbf{b}_s = i^{s-3}(b_{rr,s}, b_{\theta\theta,s}, b_{zz,s}, b_{\theta z,s}, ib_{rz,s}, ib_{r\theta,s})^T \tag{2.2}$$

It follows from Eqs. (1.4) that the components of the vectors $\mathbf{b}_s$ satisfy the following equations and boundary conditions

$$(rb_{rr,s})^{\cdot} - b_{r\theta,s} - b_{\theta\theta,s} + \delta_{s4} rb_{rz,3} = 0, \quad (rb_{r\theta,s})^{\cdot} + b_{r\theta,s} - b_{\theta\theta,s} + \delta_{s4} rb_{rz,3} = 0$$
$$(rb_{rz,s})^{\cdot} + b_{z\theta,s} + \delta_{s4} rb_{rz,3} = 0 \tag{2.3}$$

$$b_{rr,s}(r_\alpha) = 0, \quad b_{z\theta,s}(r_\alpha) = 0, \quad b_{rz,s}(r_\alpha) = 0 \tag{2.4}$$

where δ_{s4} is the Kronecker delta, and the superscipt dot denotes a derivative with respect to r. On the other hand, using the relations of the generalized Hooke's law and the Saint-Venant solutions (2.1) we have

$$b_s = C'\varepsilon_s + C'\varepsilon_s^0$$
$$\varepsilon_s = i^{s-3}\left(\frac{da_{r,s}}{dr}, \frac{a_{r,s}-a_{\theta,s}}{r}, 0, -\frac{a_{z,s}}{r}, i\frac{da_{z,s}}{dr}, i\left(\frac{da_{\theta,s}}{dr}+\frac{a_{r,s}-a_{\theta,s}}{r}\right)\right)^T$$
$$\varepsilon_3^0 = (0, 0, -r, 0, 0, 0)^T, \quad \varepsilon_4^0 = (0, 0, ia_{z,3}, ia_{\theta,3}, 0, 0)^T \tag{2.5}$$

Relations (2.2)–(2.5) lead to the integration of a system of three second-order ordinary differential equations with variable coefficients with respect to the functions $a_{r,s}$, $a_{\theta,s}$, $a_{z,s}$. This form of writing the boundary-value problems is convenient when investigating their solutions by analytical methods, for example, by the small-parameter method. For numerical integration it is more convenient to write the initial boundary-value problems in the form of Cauchy problems for a system of six first-order ordinary differential equations. To construct this system we introduce the six-coordinate vectors

$$y_s = (y_{1,s}, y_{2,s}, \ldots, y_{6,s})^T$$

where

$$y_{1,s} = a_{r,s}, \quad y_{2,s} = a_{\theta,s}, \quad y_{3,s} = a_{z,s}, \quad y_{4,s} = \frac{rb_{rr,s}}{c_{11}}, \quad y_{5,s} = \frac{rb_{r\theta,s}}{c_{11}}, \quad y_{6,s} = \frac{rb_{rz,s}}{c_{11}}$$

The initial system of three second-order ordinary differential equations can now be written in the form

$$\frac{d\mathbf{y}_s}{dr} = A\mathbf{y}_s + \mathbf{q}_s \tag{2.6}$$

The non-zero elements of the matrix A and the coefficients of the vectors $\mathbf{q}_s$ have the form

$$A_{11} = -A_{12} = -\frac{c'_{12}}{rc_{11}}, \quad A_{13} = \frac{c'_{14}}{rc_{11}}, \quad A_{14} = -A_{21} = A_{22} = \frac{1}{r}$$
$$A_{25} = -\frac{c'_{55}K_1}{r}, \quad A_{26} = A_{35} = \frac{c'_{56}K_1}{r}, \quad A_{36} = -\frac{c'_{66}K_1}{r}$$
$$A_{41} = -A_{42} = \frac{K_2}{r}, \quad A_{43} = \frac{K_4}{r}, \quad A_{44} = \frac{c'_{12}}{rc_{11}}, \quad A_{45} = \frac{1}{r}, \quad A_{5j} = -A_{4j}, \quad j = 1, 2, \ldots, 6$$
$$A_{61} = -A_{62} = \frac{K_4}{r}, \quad A_{63} = \frac{K_6}{r}, \quad A_{64} = -\frac{c'_{14}}{rc_{11}}$$
$$q_{1,3} = \frac{c'_{13}}{rc_{11}}, \quad q_{1,4} = \left(-\frac{c'_{13}}{c_{11}}a_{z,3} - \frac{c'_{14}}{c_{11}}a_{\theta,3}\right), \quad q_{3,4} = a_{r,3}$$
$$q_{4,3} = -q_{5,3} = -rK_3, \quad q_{4,4} = K_3a_{z,3} + K_4a_{\theta,3} - \frac{r}{c_{11}}b_{rz,3}$$
$$q_{5,4} = -K_3a_{z,3} - K_4a_{\theta,3} - \frac{r}{c_{11}}b_{z\theta,3}$$
$$q_{6,3} = rK_5, \quad q_{6,4} = -K_5a_{z,3} - K_6a_{\theta,3} - \frac{r}{c_{11}}b_{zz,3}$$

Here

$$K_1 = \frac{c_{11}}{c'^2_{56} - c'_{55}c'_{66}}, \quad K_2 = \frac{c_{11}c'_{22} - c'^2_{12}}{c^2_{11}}, \quad K_3 = \frac{c_{11}c'_{23} - c'_{12}c'_{13}}{c^2_{11}}$$
$$K_4 = \frac{c_{11}c'_{24} - c'_{12}c'_{44}}{c^2_{11}}, \quad K_5 = \frac{c_{11}c'_{34} - c'_{13}c'_{14}}{c^2_{11}}, \quad K_6 = \frac{c_{11}c'_{44} - c'^2_{14}}{c^2_{11}}$$

3. Methods of constructing elementary solutions and some results of a numerical analysis

It is only possible to integrate Eqs. (2.6) for arbitrary values of the parameter τ by numerical methods. However, for small values of the dimensionless parameter $\tau_0 = \tau r_2$ we can construct approximate analytical solutions by the small-parameter method. These solutions enable us, on the one hand, to obtain a clear representation of the effect of different parameters of the problems on their solution, and on the other hand, they can be used as tests for the numerical integration.

We will first construct analytical solutions by the small-parameter method. We will only outline its scheme here since it is described in detail in Ref. 4 when constructing solutions of tension torsion problems.

For small τ_0 we will seek a solution in the form

$$a_{j,s} = a_{j,s}^{(0)} + \tau_0 a_{j,s}^{(1)} + \dots \tag{3.1}$$

After some reduction we obtainfor $s = 3$

$$a_{r,3} = \frac{\nu' r^2}{2} + O(\tau_0^2), \quad a_{\theta,3} = -\frac{\nu' r^2}{2} + O(\tau_0^2), \quad a_{z,3} = \frac{\tau_0 B_1}{8c_{44}r_2}[r^3 - 3r\kappa_+(r)] + O(\tau_0^3)$$

$$b_{zz,3} = -\frac{E' r}{r_2} + O(\tau_0^2), \quad b_{rz,3} = \frac{3\tau_0 B_1}{8r_2}[r^2 - \kappa_-(r)] + O(\tau_0^3)$$

$$b_{\theta z,3} = -\frac{3\tau_0 B_1}{r_2}[3r^2 - \kappa_+(r)] + O(\tau_0^3), \quad b_{rr,3} = b_{\theta\theta,3} = b_{r\theta,3} = O(\tau_0^2)$$

for $s = 4$

$$a_{r,4} = O(\tau_0), \quad a_{\theta,4} = O(\tau_0), \quad a_{z,4} = \frac{1}{8G' r_2}[B_2 r^3 - B_3 r\kappa_+(r)] + O(\tau_0^2)$$

$$b_{rz,4} = \frac{B_3}{8}[r^2 - \kappa_-(r)] + O(\tau_0^2), \quad b_{\theta z,4} = \frac{1}{8}[B_2 r^2 - \kappa_+(r)] + O(\tau_0^2)$$

$$b_{rr,4} \simeq b_{\theta\theta,4} \simeq b_{zz,4} \simeq b_{r\theta,4} = O(\tau_0)$$

Here

$$\kappa_\pm(r) = r_1^2 + r_2^2 \pm \frac{r_1^2 r_2^2}{r^2}, \quad \nu' = \frac{c_{13}}{c_{11} + c_{12}}, \quad E' = c_{33} - 2\nu' c_{13}, \quad G' = c_{44}$$

$$B_1 = E' - 2(1+\nu')G', \quad B_2 = E' + 2\nu' G', \quad B_3 = 3E + 2\nu' G'$$

For a numerical integration of boundary-value problems (2.6) and (2.4), the solutions will be sought in the form

$$\mathbf{y}_s = \mathbf{y}_s^0 + \sum_{p=1}^{3} X_p \mathbf{y}_s^p$$

where $\mathbf{y}_s^0$, $\mathbf{y}_s^p$ are the solutions of the following Cauchy problems

$$\frac{d\mathbf{y}_s^0}{dr} = A\mathbf{y}_s^0 + \mathbf{q}_s, \quad \mathbf{y}_s^0(r_1) = (0,0,0,0,0,0)^T$$

$$\frac{d\mathbf{y}_s^p}{dr} = A\mathbf{y}_s^p, \quad \mathbf{y}_s^1(r_1) = (1,0,0,0,0,0)^T$$

$$\mathbf{y}_s^2(r_1) = (0,1,0,0,0,0)^T \quad \mathbf{y}_s^3(r_1) = (0,0,1,0,0,0)^T$$

and, in order for the solutions to satisfy the boundary conditions when $r = r_2$, the constants X_p must be found from the conditions

$$y_{l,s}(r_2) = \sum_{p=1}^{3} X_p y_{l,s}^p(r_2) + y_{l,s}^0(r_2) = 0, \quad l = 4,5,6 \tag{3.2}$$

Remark. The following first integral is obtained from the first two equations of (2.3)

$$b_{rr,3} + b_{r\theta,3} = 0$$

It follows, in turn, from this relation that the rank of the matrix of algebraic system (3.2) is equal to 2, and it turns out that the rank of the augmented matrix is also equal to 2. This gives rise to some inconvenience in the numerical integration, which is easily overcome.

As an example, consider the boundary-value problem with the following boundary conditions on the ends of the cylinder

$$z = 0: u_r = u_\theta = u_z = 0 \tag{3.3}$$

$$z = L: \sigma_{rr} = p_r, \quad \sigma_{r\theta} = p_\theta, \quad \sigma_{zz} = p_z \tag{3.4}$$

We will assume that the vector of the external forces p_r, p_θ, p_z is equivalent, in the integral sense, to the transverse forces Q_{x_1}, Q_{x_2} and the bending moments $M^*_{x_1}$, $M^*_{x_2}$. The following relations are obtained from these assumptions

$$\begin{aligned}&\int_0^{2\pi}\int_{r_1}^{r_2}(p_r + ip_\theta)e^{i(\theta+\tau L)}rdrd\theta = Q_{x_1} + iQ_{x_2}, \quad \int_0^{2\pi}\int_{r_1}^{r_2}p_z rdrd\theta = 0\\ &\int_0^{2\pi}\int_{r_1}^{r_2}p_z e^{-i(\theta+\tau L)}r^2drd\theta = iM^*_{x_1} - M^*_{x_2}, \quad \int_0^{2\pi}\int_{r_1}^{r_2}p_\theta r^2 drd\theta = 0\end{aligned} \tag{3.5}$$

After calculating σ_{rz}, $\sigma_{\theta z}$, σ_{zz}, corresponding to solution (2.1), using relations (2.2), substituting them into conditions (3.4) and subsequent integration, we obtain, taking the relations of generalized orthogonality[4] into account,

$$\begin{aligned}&dC_4 = Q_{x_1} + iQ_{x_2}, \quad dC_3 + (Ld + d')C_4 = iM^*_{x_1} - M^*_{x_2}\\ &d = 2\pi\int_{r_1}^{r_2}b_{zz,3}r^2dr, \quad d' = 2\pi\int_{r_1}^{r_2}b_{zz,4}r^2dr\end{aligned} \tag{3.6}$$

where d and d' are the elements of the stiffness matrix[4] and $d = \pi E'(r_2^4 - r_1^4)/2$, $d' = 0$ when $\tau = 0$.

We emphasise that, using Eqs. (3.6), the constants C_3 and C_4 can be determined "exactly", but the constants C_1 and C_2 can only be determined "exactly" using the solution of an infinite system of algebraic equations. A method of constructing one of the versions of this system was given earlier in Refs. 4 and 8. An asymptotic analysis of such systems shows that C_1 and C_2 are of the order of r_2/L, and hence we can put $C_1 = C_2 = 0$.

We have the following formulae for the displacements and stresses in the case of pure bending

$$u_{x_1} = -\frac{M^*_{x_1}}{d}(a_{r,3} - a_{\theta,3})\sin 2\psi - \frac{M^*_{x_2}}{d}(z^2 + 2a_{r,3}\cos^2\psi + 2a_{\theta,3}\sin^2\psi)$$

$$u_{x_2} = -\frac{M^*_{x_1}}{d}(z^2 + 2a_{r,3}\sin^2\psi + 2a_{\theta,3}\cos^2\psi) - \frac{M^*_{x_2}}{d}(a_{r,3} - a_{\theta,3})\sin 2\psi$$

$$u_{x_3} = \frac{2M^*_{x_1}}{d}(zr\sin\psi - a_{z,3}\cos\psi) + \frac{2M^*_{x_2}}{d}(zr\cos\psi + a_{z,3}\sin\psi)$$

$$\sigma_n = -2b_{n,3}\left(\frac{M^*_{x_1}}{d}\sin\psi + \frac{M^*_{x_2}}{d}\cos\psi\right), \quad n = 1, 2, 3, 4$$

$$\sigma_n = 2b_{n,3}\left(-\frac{M^*_{x_1}}{d}\cos\psi + \frac{M^*_{x_2}}{d}\sin\psi\right), \quad n = 5, 6$$

If we put $r = 0$ in the formulae for the displacements and take into account the fact that, for a continuous cylinder $a_{r,3} = a_{\theta,3} = a_{z,3} = 0$ on its axis, we obtain the classical equations of the curved axis of a rod

$$u^0_{x_1} = -\frac{M^*_{x_2}}{d}z^2, \quad u^0_{x_2} = -\frac{M^*_{x_1}}{d}z^2$$

In the case of bending by transverse forces we have

$$u_{x_1} = \frac{Q_{x_1}}{d}\left[\frac{z^3}{3} - L'z^2 + (z-L')\upsilon_3^{(1)} - \upsilon_4^{(2)}\right] - \frac{Q_{x_2}}{d}[(z-L')\upsilon_3^{(2)} - \upsilon_4^{(1)}]$$

$$u_{x_2} = \frac{Q_{x_1}}{d}[(z-L')\upsilon_3^{(2)} - \upsilon_4^{(3)}] - \frac{Q_{x_2}}{d}\left[\frac{z^3}{3} - L'z^2 + (z-L')\upsilon_3^{(3)} + \upsilon_4^{(2)}\right]$$

$$\upsilon_s^{(1)} = 2(a_{r,s}\cos^2\psi + a_{\theta,s}\sin^2\psi), \quad \upsilon_s^{(2)} = (a_{r,s} - a_{\theta,s})\sin 2\psi$$

$$\upsilon_s^{(3)} = 2(a_{r,s}\sin^2\psi - a_{\theta,s}\cos^2\psi); \quad s = 3, 4$$

$$u_{x_3} = -\frac{Q_{x_1}}{d}\left[r\left(\frac{z^2}{2} - L'z\right)\cos\psi + 2(z-L')a_{z,3}\sin\psi + 2a_{z,4}\cos\psi\right] +$$

$$+\frac{Q_{x_2}}{d}\left[r\left(\frac{z^2}{2} - L'z\right)\sin\psi - 2(z-L')a_{z,3}\cos\psi + 2a_{z,4}\sin\psi\right]$$

$$\sigma_n = 2b_{n,3}(z-L')R - 2b_{n,4}S, \quad n = 1, 2, 3, 4$$

$$\sigma_n = -2b_{n,3}(z-L')S - 2b_{n,4}R, \quad n = 5, 6$$

$$R = \frac{Q_{x_1}}{d}\cos\psi - \frac{Q_{x_2}}{d}\sin\psi, \quad S = \frac{Q_{x_1}}{d}\sin\psi + \frac{Q_{x_2}}{d}\cos\psi, \quad L' = L + \frac{d'}{d}$$

We have the following equivalence relations for n

$$1 \sim rr, \quad 2 \sim \theta\theta, \quad 3 \sim zz, \quad 4 \sim z\theta, \quad 5 \sim rz, \quad 6 \sim r\theta$$

The equations of the curved axis in this case have the form

$$u_{x_1}^0 = \frac{Q_{x_1}}{d}\left(\frac{z^3}{3} - L'z^2\right), \quad u_{x_2}^0 = -\frac{Q_{x_2}}{d}\left(\frac{z^3}{3} - L'z^2\right) \tag{3.7}$$

When $\alpha = 0$, $\alpha = \pi/2$ ($\alpha = \mathrm{arctg}\tau_0$), i.e., when helical anisotropy degenerates into cylindrical anisotropy, $L' = L$.

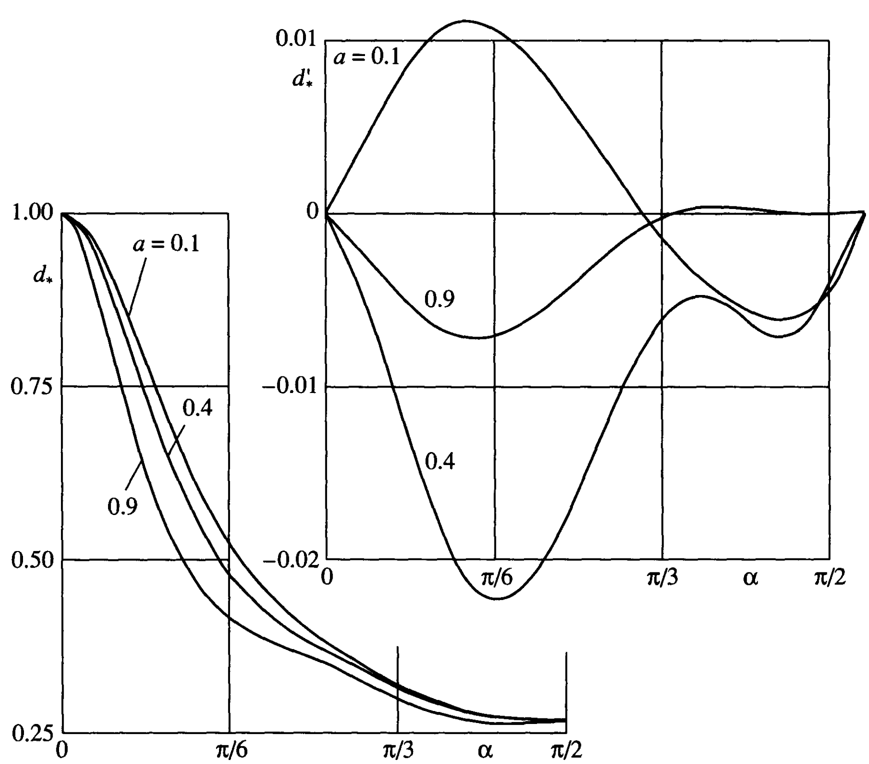

Fig. 1.

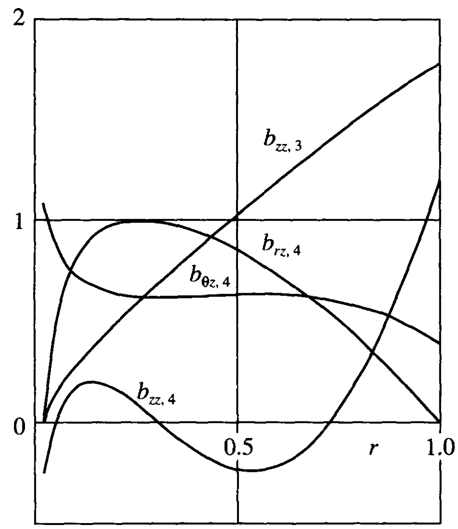

Fig. 2.

In conclusion we will present some results of a numerical analysis of the problem. All the calculations were carried out for a cylinder made of composite material, the average elastic characteristics of which have the following values in a Frenet basis

$$E = 3.622 \cdot 10^{11}\ \text{Pa},\ E' = 1.557 \cdot 10^{11}\ \text{Pa},\ G' = 1.053 \cdot 10^{11}\ \text{Pa},\ \nu = 0.31,\ \nu' = 0.336$$

The moduli c_{ij}' were calculated using well-known formulae.

By numerical integration we obtained the following dependences of the normalized elements of the stiffness matrix

$$d_* = d/d_0, \quad d'_* = d'/(r_2 d_0)\ (d_0 = E' r_2^4 (1 - a^4)/2)$$

on the parameter $\alpha = \text{arctg}\tau_0$ for different values of the parameter $a = r_1/r_2$ (Fig. 1).

In Fig. 2 we show graphs of $b_{zz,3}$, $b_{zz,4}$ and $b_{rz,4}$, $b_{\theta z,4}$, corresponding to $\alpha = \pi/6$, which illustrate the distribution of the normal stresses σ_{zz} and the torsional stresses σ_{rz}, $\sigma_{\theta z}$ respectively over the cross-section.

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